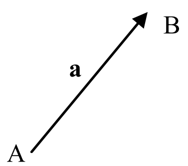
VECTORS

Representation of vectors :

Geometrically a vector is represent by a line segment. For example, $\mathbf{a} = \overrightarrow{AB}$. Here A is called the initial point and B, the terminal point or tip.

Magnitude or modulus of **a** is expressed as

$|\mathbf{a}| = |\overrightarrow{AB}| = AB.$



Types of Vector:

- **Zero or null vector :** A vector whose magnitude is zero is called zero or null vector and it is represented by $\vec{O}$.
- **Unit vector :** A vector whose modulus is unity, is called a unit vector. The unit vector in the direction of a vector **a** is denoted by $\hat{a}$, read as "a cap". Thus, $|\hat{a}| = 1$.

  $$\hat{a} = \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\text{Vector a}}{\text{Magnitude of a}}$$

- **Like and unlike vectors :** Vectors are said to be like when they have the same sense of direction and unlike when they have opposite directions.
- **Collinear or parallel vectors :** Vectors having the same or parallel supports are called collinear or parallel vectors.
- **Co-initial vectors :** Vectors having the same initial point are called co-initial vectors.
- **Coplanar vectors :** A system of vectors is said to be coplanar, if their supports are parallel to the same plane.

  Two vectors having the same initial point are always coplanar but such three or more vectors may or may not be coplanar.
- **Negative of a vector :** The vector which has the same magnitude as the vector **a** but opposite direction, is called the negative of **a** and is denoted by –**a**. Thus, if $\overrightarrow{PQ} = \mathbf{a}$, then $\overrightarrow{QP} = -\mathbf{a}$.

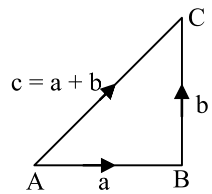
Properties of vectors :

(i) **Addition of vectors**

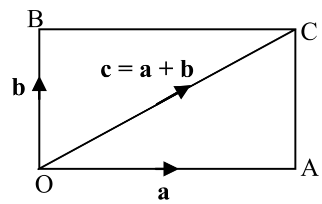
- **Triangle law of addition :** If in a ΔABC,

  $\overrightarrow{AB} = \mathbf{a}$, $\overrightarrow{BC} = \mathbf{b}$ and $\overrightarrow{AC} = \mathbf{c}$, **then**

  $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ i.e., $\mathbf{a} + \mathbf{b} = \mathbf{c}$



- **Parallelogram law of addition :** If in a parallelogram OACB, $\overrightarrow{OA} = a$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$



  Then $\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC}$ i.e., $\mathbf{a} + \mathbf{b} = \mathbf{c}$, where OC is a diagonal of the parallelogram OABC.
- **Addition in component form :** If the vectors are defined in terms of **i**, **j**, and **k**, i.e.,

  if $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, then their sum is defined as

  $\mathbf{a} + \mathbf{b} = (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j} + (a_3 + b_3)\mathbf{k}.$

Properties of vector addition :

Vector addition has the following properties.

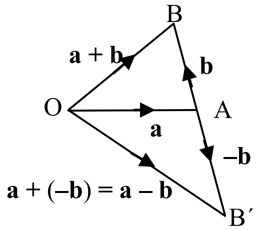
- **Binary operation :** The sum of two vectors is always a vector.
- **Commutativity :** For any two vectors **a** and **b**, $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$.
- **Associativity :** For any three vectors **a, b** and **c**, $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
- **Identity :** Zero vector is the identity for addition. For any vector **a**, $\mathbf{0} + \mathbf{a} = \mathbf{a} = \mathbf{a} + \mathbf{0}$
- **Additive inverse :** For every vector **a** its negative vector –**a** exits such that $\mathbf{a} + (-\mathbf{a}) = (-\mathbf{a}) + \mathbf{a} = \mathbf{0}$ i.e., (–**a**) is the additive inverse of the vector **a**.

Subtraction of vectors :

If **a** and **b** are two vectors, then their subtraction **a – b** is defined as **a – b** = **a** + (–**b**) where –**b** is the negative of **b** having magnitude equal to that of **b** and direction opposite to **b**. If

$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

Then $\mathbf{a} - b = (a_1 - b_1)\mathbf{i} + (a_2 - b_2)\mathbf{j} + (a_3 - b_3)\mathbf{k}$.



Properties of vector subtraction :

(i) $\mathbf{a} - \mathbf{b} \neq \mathbf{b} - \mathbf{a}$

(ii) $(\mathbf{a} - \mathbf{b}) - \mathbf{c} \neq \mathbf{a} - (\mathbf{b} - \mathbf{c})$

(iii) Since any one side of a triangle is less than the sum and greater than the difference of the other two sides, so for any two vectors a and b, we have

(a) $|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$

(b) $|\mathbf{a} + \mathbf{b}| \geq |\mathbf{a}| - |\mathbf{b}|$

(c) $|\mathbf{a} - \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$

(d) $|\mathbf{a} - \mathbf{b}| \geq |\mathbf{a}| - |\mathbf{b}|$

Multiplication of a vector by a scalar : If **a** is a vector and m is a scalar (i.e., a real number) then m**a** is a vector whose magnitude is m times that of **a** and whose direction is the same as that of **a**, if m is positive and opposite to that of **a**, if m is negative.

Properties of Multiplication of vector by a scalar : The following are properties of multiplication of vectors by scalars, for vector **a, b** and scalars m, n.

(i) $m(-\mathbf{a}) = (-m)\mathbf{a} = -(m\mathbf{a})$

(ii) $(-m)(-\mathbf{a}) = m\mathbf{a}$

(iii) $m(n\mathbf{a}) = (mn)\mathbf{a} = n(m\mathbf{a})$

(iv) $(m + n)\mathbf{a} = m\mathbf{a} + n\mathbf{a}$

(v) $m(\mathbf{a} + \mathbf{b}) = m\mathbf{a} + m\mathbf{b}$

Position vector :

- $\overrightarrow{AB}$ **in terms of the position vectors of points A and B** : If **a** and **b** are position vectors of points A and B respectively. Then, $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$

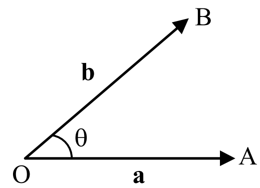
$\therefore$ $\overrightarrow{AB}$ = (Position vector of B)– (Position vector of A)

$= \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{b} - \mathbf{a}$

- **Position vector of a dividing point :** The position vectors of the points dividing the line AB in the ratio m : n internally or externally are $\dfrac{m\mathbf{b} + n\mathbf{a}}{m + n}$ or $\dfrac{m\mathbf{b} - n\mathbf{a}}{m - n}$.

Scalar or Dot product

Scalar or Dot product of two vectors : If **a** and **b** are two non-zero vectors and θ be the angle between them, then their scalar product (or dot product) is denoted by **a** . **b** and is defined as the scalar $|\mathbf{a}|\,|\mathbf{b}|\cos\theta$, where $|\mathbf{a}|$ and $|\mathbf{b}|$ are modulii of **a** and **b** respectively and $0 \leq \theta \leq \pi$. Dot product of two vectors is a scalar quantity.



Angle between two vectors : If **a, b** be two vectors inclined at an angle θ, then $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$

$\Rightarrow \cos\theta = \dfrac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$

$\Rightarrow \theta = \cos^{-1}\left(\dfrac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\right)$

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$; then

$$\theta = \cos^{-1}\left(\frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}\right)$$

Properties of scalar product

- **Commutativity :** The scalar product of two vector is commutative i.e., **a . b = b . a**
- **Distributivity of scalar product over vector addition :** The scalar product of vectors is distributive over vector addition i.e.,

  (a) **a . (b + c) = a . b + a . c**, (Left distributivity)

  (b) (**b + c**) . **a** = **b . a + c . a**, (Right distributivity)
- Let **a** and **b** be two non-zero vectors **a . b** = 0

  $\Leftrightarrow \mathbf{a} \perp \mathbf{b}$.

  As **i, j, k** are mutually perpendicular unit vectors along the coordinate axes, therefore,

  i . j = **j . i** = 0; **j . k** = **k . j** = 0; **k . i** = **i . k** = 0.
- For any vector $\mathbf{a} \cdot \mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$.

  As **i. j. k** are unit vectors along the co-ordinate axes, therefore $\mathbf{i} \cdot \mathbf{i} = |\mathbf{i}|^2 = 1$, $\mathbf{j} \cdot \mathbf{j} = |\mathbf{j}|^2 = 1$ and $\mathbf{k} \cdot \mathbf{k} = |\mathbf{k}|^2 = 1$
- If m, n are scalars and **a . b** be two vectors, then $m\mathbf{a} \cdot n\mathbf{b} = mn(\mathbf{a} \cdot \mathbf{b}) = (mn\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (mn\mathbf{b})$
- For any vectors **a** and **b**, we have

  (a) $\mathbf{a} \cdot (-\mathbf{b}) = -(\mathbf{a} \cdot \mathbf{b}) = (-\mathbf{a}) \cdot \mathbf{b}$

  (b) $(-\mathbf{a}) \cdot (-\mathbf{b}) = \mathbf{a} \cdot \mathbf{b}$
- For any two vectors **a** and **b**, we have

  (a) $|\mathbf{a} + \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2\mathbf{a}.\mathbf{b}$

  (b) $|\mathbf{a} - \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\mathbf{a}.\mathbf{b}$

(c) $(\mathbf{a}+\mathbf{b}) \cdot (\mathbf{a}-b) = |\mathbf{a}|^2 - |\mathbf{b}|^2$

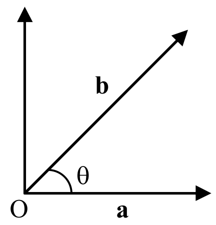
(d) $|\mathbf{a}+\mathbf{b}| = |\mathbf{a}| + |\mathbf{b}| \Rightarrow \mathbf{a} \,|\,|\, \mathbf{b}$

(e) $|\mathbf{a}+\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 \Rightarrow \mathbf{a} \perp \mathbf{b}$

(f) $|\mathbf{a}+\mathbf{b}| = |\mathbf{a}-\mathbf{b}| \Rightarrow \mathbf{a} \perp \mathbf{b}$

Vector or Cross product

Vector product of two vectors : Let **a**, **b** be two non-zero, non-parallel vectors.



Then $\mathbf{a} \times \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \sin\theta\ \hat{\eta}$, and $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}| \sin\theta$, where θ is the angle between **a** and **b**, $\hat{\eta}$ is a unit vector perpendicular to the plane of **a** and **b** such that **a**, **b**, $\hat{\eta}$ form a right-handed system.

Properties of vector product :

- Vector product is not commutative i.e., if **a** and **b** are any two vectors, then $\mathbf{a} \times \mathbf{b} \neq \mathbf{b} \times \mathbf{a}$, however, $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$
- If **a**, **b** are two vectors and m, n are scalars, then $m\mathbf{a} \times n\mathbf{b} = mn(\mathbf{a} \times \mathbf{b}) = m(\mathbf{a} \times n\mathbf{b}) = n(m\mathbf{a} \times \mathbf{b})$.
- Distributivity of vector product over vector addition. Let **a**, **b**, **c** be any three vectors. Then
  (a) $\mathbf{a} \times (\mathbf{b}+\mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$ (left distributivity)
  (b) $(\mathbf{b}+\mathbf{c}) \times \mathbf{a} = \mathbf{b} \times \mathbf{a} + \mathbf{c} \times \mathbf{a}$ (Right disribustivity)
- For any three vectors **a**, **b**, **c** we have $\mathbf{a} \times (\mathbf{b}-\mathbf{c}) = \mathbf{a} \times \mathbf{b} - \mathbf{a} \times \mathbf{c}$.
- The vector product of two non-zero vectors is zero vector iff they are parallel (Collinear) i.e.,
  $\mathbf{a} \times \mathbf{b} = 0 \Leftrightarrow \mathbf{a} \,|\,|\, \mathbf{b}$, **a, b** are non-zero vectors.
  It follows from the above property that $\mathbf{a} \times \mathbf{a} = 0$ for every non-zero vector **a**, which in turn implies that $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$.
- Vector product of orthonormal triad of unit vectors **i, j, k** using the definition of the vector product, we obtain $\mathbf{i} \times \mathbf{j} = \mathbf{k}, \mathbf{j} \times \mathbf{k} = \mathbf{i}, \mathbf{k} \times \mathbf{i} = \mathbf{j}, \mathbf{j} \times \mathbf{i} = -\mathbf{k}, \mathbf{k} \times \mathbf{j} = -\mathbf{i}, \mathbf{i} \times \mathbf{k} = -\mathbf{j}$.

Vector product in terms of components :

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$.

Then, $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

Angle between two vectors :

If θ is the angle between **a** and **b** then $\sin\theta = \dfrac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|\,|\mathbf{b}|}$

Right handed system of vectors : Three mutually perpendicular vectors **a**, **b**, **c** form a right handed system of vector iff $\mathbf{a} \times \mathbf{b} = \mathbf{c}, \mathbf{b} \times \mathbf{c} = \mathbf{a}, \mathbf{c} \times \mathbf{a} = \mathbf{b}$

Left handed system of vectors : The vectors **a, b, c** mutually perpendicular to one another form a left handed system of vector iff $\mathbf{c} \times \mathbf{b} = \mathbf{a}, \mathbf{a} \times \mathbf{c} = \mathbf{b}, \mathbf{b} \times \mathbf{a} = \mathbf{c}$.

Area of parallelogram and triangle :

- The area of a parallelogram with adjacent sides **a** and **b** is $|\mathbf{a} \times \mathbf{b}|$.
- The area of a plane quadrilateral ABCD is $\frac{1}{2}|\overrightarrow{AC} \times \overrightarrow{BD}|$, where AC and BC are its diagonals.
- The area of a triangle ABC is $\frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$ or $\frac{1}{2}|\overrightarrow{BC} \times \overrightarrow{BA}|$ or $\frac{1}{2}|\overrightarrow{CB} \times \overrightarrow{CA}|$

Scalar triple product

Scalar triple product of three vectors : If **a**, **b**, **c** are three vectors, then scalar triple product is defined as the dot product of two vectors **a** and $\mathbf{b} \times \mathbf{c}$. It is generally denoted by $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ or $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$.

Properties of scalar triple product : If **a**, **b**, **c** are cyclically permuted, the value of scalar triple product remains the same. i.e.,

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b}$$

or $$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = [\mathbf{b}\ \mathbf{a}\ \mathbf{c}] = [\mathbf{c}\ \mathbf{a}\ \mathbf{b}]$$

Vector triple product

Let **a**, **b**, **c** be any three vectors, then the vectors $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ and $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ are called vector triple product of **a**, **b**, **c**.

Thus, $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

Properties of vector triple product :

- The vector triple product $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is a linear combination of those two vectors which are within brackets.
- The vector $\mathbf{r} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is perpendicular to **a** and lies in the plane of **b** and **c**.
- The formula $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$ is true only when the vector outside the bracket is on the left most side. If it is not, we first shift on left by using the properties of cross product and then apply the same formula.
  Thus, $(\mathbf{b} \times \mathbf{c}) \times \mathbf{a} = -\{\mathbf{a} \times (\mathbf{b} \times \mathbf{c})\}$
  $= \{(\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}\}$
  $= (\mathbf{a} \cdot \mathbf{b})\mathbf{c} - (\mathbf{a} \cdot \mathbf{c})\mathbf{b}$
- Vector triple product is a vector quantity.
- $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$